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TOPICAL ISSUE

Necessary Extremum Conditions and the Neustadt–Eaton Method in the Time-Optimal Control Problem for a Group of Nonsynchronous Oscillators

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Abstract—The time-optimal control problem for an arbitrary number of nonsynchronous oscillators with a limited scalar control is considered. An analytical investigation of the problem is performed. The property of strong accessibility and global controllability is proved, and a program control is found that brings the system from the origin to a fixed point in the shortest time. Trajectories satisfying both the motion equations of the system and the additional conditions based on the matrix nondegeneracy conditions of the relay control have been found for bringing a group of oscillators to the origin. Two classification methods of trajectories according to the number of control switchings are compared: the one based on the necessary extremum conditions and the Neustadt–Eaton numerical algorithm.

Keywords: Pontryagin's maximum principle, optimal control, nonsynchronous oscillators, Neustadt–Eaton numerical algorithm, strong accessibility, global controllability, geometric control theory

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1. INTRODUCTION

In the 1960s, American mathematicians Eaton and Neustadt proposed an iterative method of rotating the supporting hyperplane [1, 2] to find a solution for the time-optimal control problem based on the ideas of convex optimization. The iterative algorithm is applicable in case of the convex reachable set and allows to compute the initial vector of the adjoint system, which is used to determine the optimal trajectory. The drawback of this approach is that an infinite number of non-collinear initial vectors of the adjoint system can correspond to one initial state of the system [3], which significantly affects on the convergence of the algorithm. Article [4] emphasizes that convergence of the above algorithm is possible only when the reachable set has a strict convexity property. Checking this property is not possible for nonlinear problems. In turn, for controllable linear systems, the convexity of the reachable set follows from the compactness and convexity of the admissible control set [5]. The linear time-optimal control problem for a group of nonsynchronous oscillators with a convex set of admissible controls has the strict convexity property of the reachable set. A technique for choosing the step of the Neustadt–Eaton algorithm, which is responsible for the rate of convergence, is proposed in [6]. A proposal to improve the convergence of the method was considered in [7]. Another iterative minimax method based on the Minkowski function is presented in [8], where the interpretation of the above methods in the adjoint space is given. An iterative algorithm for computing time-optimal solutions for linear systems based on the maximum principle is given in [9]. The convergence of numerical methods in optimization problems was the subject of B.T. Polyak's early works, e.g. [10].

Other methods for finding approximate solutions in the time-optimal control problems for linear systems are diverse and particularly interesting due to a variety of used approaches. Real-time control finding methods based on preliminary approximation of reachable sets and the initial vector computation of the adjoint system are considered in [11]. Simplex covers of the reachable set convex hull are at the core of the iterative algorithm, which is used in this paper to find the minimum time of motion of linear [12] and nonlinear [13] controllable systems, such that the origin will belong to the reachable set boundary. The invariant ellipsoid technique developed by B.T. Polyak and his co-authors [14] is the basis for constructing efficient estimates of the reachable set of systems with external perturbations.

For the suppression problem of an arbitrary number of linear oscillators with a scalar control in [15], asymptotic optimal control has been derived in the synthesis form, where several approaches are combined. The first idea is using the normal to the approximate reachable set as the initial vector for the adjoint variables for large energies. A small neighborhood of the origin can be reached by applying a control with a reduced upper bound. Finally, the generalized Lyapunov function method is applied to construct the synthesis in the neighborhood of the terminal state. A control structure represented by a single external constrained force is investigated for a platform with an arbitrary number of linear oscillators [16].

The paper is devoted to the problem of limited scalar control for the group of nonsynchronous oscillators with time criterion, for which, to the authors' knowledge, the Neustadt–Eaton algorithm has not been applied. The idea arose to compare the analytical results obtained earlier for two nonsynchronous oscillators at the initial value plane classification of the first oscillator by control classes [17] and to extend the algorithm to a group of a larger number of oscillators [18].

The paper consists of eight sections, including an introduction and conclusion, and has the following structure. A formulation of the time-optimal control problem for a system of N nonsynchronous oscillators with constrained scalar control is given in Section 2. Section 3 deals with the controllability of the system and connects the controls that bring the system to the boundary of the reachable set to the ones that bring the system to the origin. An optimal control containing unknown constants from the fundamental solution of the adjoint system is obtained in Section 4 according to the Pontryagin's maximum principle. In section 5, the necessary extremum conditions are given in the form of nonlinear matrix equations. Section 6 describes the application of the Neustadt–Eaton numerical procedure to find the initial vector of the adjoint system to construct an approximated optimal control. The obtained results are illustrated by modeling in 7. After that, plans for further work are formulated in 8.

2. OPTIMAL CONTROL PROBLEM STATEMENT

A linear optimal control problem for N nonsynchronous oscillators coupled by a constrained scalar control [18] is considered. The motion of this control system is described by the following equations:

$$
\begin{gathered}
\begin{cases} \dot{q}_i(t) = p_i(t), \\ \dot{p}_i(t) = -\omega_i^2 q_i(t) + u(t), \end{cases} \quad i = 1, \ldots, N. \\
\omega_j \neq \omega_k, \ \forall j \neq k, \quad j, k = 1, \ldots, N \\
\mathbf{x}(t) = (q_1(t), p_1(t), \ldots, q_N(t), p_N(t))^{\mathrm{T}} \in \mathbb{R}^{2N}.
\end{gathered} \tag{1}
$$

The components $q_i(t), p_i(t)$ of the state vector $\mathbf{x}(t)$ are the coordinate and momentum of the ith oscillator with a natural frequency ω_i, $i = 1, \ldots, N$. The range of control values $\mathbb{U}$ is given by the line segment:

$$u(t) \in [-\varepsilon, \varepsilon] = \mathbb{U}, \quad u(t) \in \mathcal{L}^{\infty}. \tag{2}$$

The system (1) can be rewritten in matrix form

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + Bu(t),$$

where A is the system matrix, B is the control matrix:

$$A = \begin{pmatrix} A_1 & 0 & 0 & \dots & 0 \\ 0 & A_2 & 0 & \dots & 0 \\ 0 & 0 & A_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & A_N \end{pmatrix}, \quad A_i = \begin{pmatrix} 0 & 1 \\ -\omega_i^2 & 0 \end{pmatrix}, \; i = 1, \dots, N, \; B = \begin{pmatrix} 0 \\ 1 \\ \dots \\ 0 \\ 1 \end{pmatrix}.$$

It is required to bring the system (1) from the given position to the origin

$$\begin{gathered} \mathbf{x}(0) = \mathbf{x}_0 = (q_1^*, p_1^*, \dots, q_N^*, p_N^*)^{\mathrm{T}}, \\ \mathbf{x}(T^0) = \mathbf{x}_{T^0} = (0, 0, \dots, 0, 0)^{\mathrm{T}}. \end{gathered} \tag{3}$$

The system motion time T^0 is a criterion for the time-optimal control problem

$$T^0[u] = \int_0^{T^0} dt \to \min_{u(\cdot) \in \mathbb{U}}. \tag{4}$$

For the system (1)–(3), the controllability problem will be investigated in the next section using the geometric control theory [19].

3. CONTROLLABILITY PROBLEM

To study the controllability property of the system (1) with constraints (2), let us rewrite it as a vector field:

$$\mathcal{F}(\mathbf{x}, u) = \{f_1 + uf_2 \mid u \in \mathbb{U}\}, \tag{5}$$

$$f_1 = p_1 \frac{\partial}{\partial q_1} + \dots + p_N \frac{\partial}{\partial q_N} - \omega_1^2 q_1 \frac{\partial}{\partial p_1} - \dots - \omega_N^2 q_N \frac{\partial}{\partial p_N} = \sum_{i=1}^{N} p_i \frac{\partial}{\partial q_i} - \omega_i^2 q_i \frac{\partial}{\partial p_i}, \tag{6}$$

$$f_2 = \frac{\partial}{\partial p_1} + \dots + \frac{\partial}{\partial p_N} = \sum_{i=1}^{N} \frac{\partial}{\partial p_i}. \tag{7}$$

Definition 1. A linear system $(A, B, \mathbb{U})$ has the strong accessibility property if the reachable set at a non-zero time instant has a non-empty interior from any initial state of the system.

The strong accessibility property for the system (1)–(2) is obtained from the Sussmann–Jurdjevic [20] theorem.

Theorem 1 (Sussmann–Jurdjevic). *The analytic system $\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), u(t))$ has the strong accessibility property from a point $\mathbf{x}$ if and only if an ideal of the Lie algebra produced by the system coincides with the dimension of the state space*

$$dim \mathcal{L}_0(\mathbf{x}) = 2N. \tag{8}$$

Lemma 1. *The system* (1) *with constraints* (2) *is strongly accessible.*

Proof. The computation of all nonzero vector fields is given to determine the ideal dimension of the Lie algebra.

$$f_3 = [f_1, f_2] = -\begin{pmatrix} F_1 & 0 & 0 & \dots & 0 \\ 0 & F_2 & 0 & \dots & 0 \\ 0 & 0 & F_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & F_N \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ \dots \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ \dots \\ -1 \\ 0 \end{pmatrix} = -\sum_{i=1}^{N} \frac{\partial}{\partial q_i},$$

where the notations are introduced:

$$F_i = \begin{pmatrix} 0 & 1 \\ -\omega_i^2 & 0 \end{pmatrix}, \ i = 1, \dots, N.$$

$$f_4 = [f_1, f_3] = -\begin{pmatrix} F_1 & 0 & 0 & \dots & 0 \\ 0 & F_2 & 0 & \dots & 0 \\ 0 & 0 & F_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & F_N \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ \dots \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -\omega_1^2 \\ \dots \\ 0 \\ -\omega_N^2 \end{pmatrix} = -\sum_{i=1}^{N} \omega_i^2 \frac{\partial}{\partial p_i},$$

$$f_5 = [f_1, f_4] = -\begin{pmatrix} F_1 & 0 & 0 & \dots & 0 \\ 0 & F_2 & 0 & \dots & 0 \\ 0 & 0 & F_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & F_N \end{pmatrix} \begin{pmatrix} 0 \\ -\omega_1^2 \\ \dots \\ 0 \\ -\omega_N^2 \end{pmatrix} = \begin{pmatrix} \omega_1^2 \\ 0 \\ \dots \\ \omega_N^2 \\ 0 \end{pmatrix} = \sum_{i=1}^{N} \omega_i^2 \frac{\partial}{\partial q_i},$$

$$f_6 = [f_1, f_5] = -\begin{pmatrix} F_1 & 0 & 0 & \dots & 0 \\ 0 & F_2 & 0 & \dots & 0 \\ 0 & 0 & F_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & F_N \end{pmatrix} \begin{pmatrix} \omega_1^2 \\ 0 \\ \dots \\ \omega_N^2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \omega_1^4 \\ \dots \\ 0 \\ \omega_N^4 \end{pmatrix} = \sum_{i=1}^{N} \omega_i^4 \frac{\partial}{\partial p_i},$$

$$\dots$$

$$f_{2N} = [f_1, f_{2N-1}] = (-1)^{N-1} \sum_{i=1}^{N} \omega_i^{2N-2} \frac{\partial}{\partial p_i}, \ N \geqslant 1,$$

$$f_{2N+1} = [f_1, f_{2N}] = (-1)^{N} \sum_{i=1}^{N} \omega_i^{2N-2} \frac{\partial}{\partial q_i}, \ N \geqslant 1.$$

$2N$ vector fields $\{f_2, f_3, \dots, f_{2N}, f_{2N+1}\}$ are linearly independent, which follows from the determinant

$$\det \begin{pmatrix} 0 & -1 & 0 & \omega_1^2 & \dots & 0 & (-1)^N \omega_1^{2N-2} \\ 1 & 0 & -\omega_1^2 & 0 & \dots & (-1)^{N-1}\omega_1^{2N-2} & 0 \\ 0 & -1 & 0 & \omega_2^2 & \dots & 0 & (-1)^N \omega_2^{2N-2} \\ 1 & 0 & -\omega_2^2 & 0 & \dots & (-1)^{N-1}\omega_2^{2N-2} & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & -1 & 0 & \omega_N^2 & \dots & 0 & (-1)^N \omega_N^{2N-2} \\ 1 & 0 & -\omega_N^2 & 0 & \dots & (-1)^{N-1}\omega_N^{2N-2} & 0 \end{pmatrix}, \quad (9)$$

which is reduced to the determinant of the block-diagonal matrix by elementary transformations

$$\det \begin{pmatrix} W & 0 \\ 0 & W \end{pmatrix} = (\det(W))^2 = \left(\prod_{1 \leqslant i < j \leqslant N} (\omega_j^2 - \omega_i^2) \right)^2 \neq 0 \tag{10}$$

due to the system (1). This completes the proof of the lemma. During the calculation of the determinant a matrix was introduced

$$W = \begin{pmatrix} 1 & \omega_1^2 & \dots & \omega_1^{2N-2} \\ 1 & \omega_2^2 & \dots & \omega_2^{2N-2} \\ 1 & \omega_3^2 & \dots & \omega_3^{2N-2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_N^2 & \dots & \omega_N^{2N-2} \end{pmatrix},$$

which coincides with the Vandermonde matrix, whose determinant property was used in (10) [16].

Definition 2. A linear system $(A, B, \mathbb{U})$ is called globally null controllable, if for any initial condition $\mathbf{x}_0 \in \mathbb{R}^{2N}$ there exists a control $u \in \mathbb{U}$ such that the corresponding trajectory reaches the point $\mathbf{x}(T^0) = 0$ for some T^0.

Globally null controllability of a linear system with control constraints is proved by a theorem from [21].

Theorem 2 (Lasalle, Conti). *Autonomous system* $(A, B, \mathbb{U})$ *with* $\mathbb{U} \in R^m$ *and* $\operatorname{int}\mathbb{U} \neq \varnothing$ *is globally null controllable if and only if:*

1) $\operatorname{rank}[B, AB, \dots, A^{n-1}B] = n$.

2) $\operatorname{Re}(\lambda_i) \leqslant 0$ *for every eigenvalue* λ_i *of the matrix* A.

Lemma 2. *The system* (1) *with constraint* (2) *is globally controllable.*

Proof. The first condition of the Theorem 2 is equivalent to the condition (8) of the Theorem 1. The equality $\operatorname{Re}(\lambda_i) = 0$ is satisfied for the system (1). Thus, the Theorem 2 is satisfied and the system (1) is globally null controllable.

Let us prove the global controllability property by a certain construction, using the general solution of the differential equation (1) with initial state $(q_1^*, p_1^*, \dots, q_N^*, p_N^*)$.

$$\begin{cases} q_i(t) = \dfrac{p_i^*}{\omega_i} \sin(\omega_i t) + q_i^* \cos(\omega_i t) + \dfrac{1}{\omega_i} \displaystyle\int_0^t \sin(\omega_i (t - \tau)) u(\tau) d\tau, \\ p_i(t) = p_i^* \cos(\omega_i t) - q_i^* \omega_i \sin(\omega_i t) + \displaystyle\int_0^t \cos(\omega_i (t - \tau)) u(\tau) d\tau, \end{cases} \quad i = 1, \dots, N. \tag{11}$$

Null controllability provides the existence of a control $u(t)$ such that at some instant t_1 the state vector $\mathbf{x}(t_1)$ coincides with the origin.

$$\begin{cases} -\dfrac{p_i^*}{\omega_i} \sin(\omega_i t_1) - q_i^* \cos(\omega_i t_1) = \dfrac{1}{\omega_i} \displaystyle\int_0^{t_1} \sin(\omega_i (t_1 - \tau)) u(\tau) d\tau, \\ -p_i^* \cos(\omega_i t_1) + q_i^* \omega_i \sin(\omega_i t_1) = \displaystyle\int_0^{t_1} \cos(\omega_i (t_1 - \tau)) u(\tau) d\tau, \end{cases} \quad i = 1, \dots, N. \tag{12}$$

Let us show the relation between the control $u(t)$ and the control $u_0(t)$ that brings the system from the origin to a point $(q_1^*, -p_1^*, \dots, q_N^*, -p_N^*)$ in time t_0. This solution is represented in the following form

$$\begin{cases} q_i^* = \dfrac{1}{\omega_i} \displaystyle\int_0^{t_0} \sin(\omega_i(t_0 - \tau)) u_0(\tau) d\tau, \\ -p_i^* = \displaystyle\int_0^{t_0} \cos(\omega_i(t_0 - \tau)) u_0(\tau) d\tau, \end{cases} \quad i = 1, \dots, N. \tag{13}$$

Substituting (13) into (12), we obtain:

$$\begin{cases} \dfrac{1}{\omega_i} \displaystyle\int_0^{t_0} \sin(\omega_i(t_1 - t_0 + \tau)) u_0(\tau) d\tau = \dfrac{1}{\omega_i} \displaystyle\int_0^{t_1} \sin(\omega_i(t_1 - \tau)) u(\tau) d\tau, \\ \displaystyle\int_0^{t_0} \cos(\omega_i(t_1 - t_0 + \tau)) u_0(\tau) d\tau = \displaystyle\int_0^{t_1} \cos(\omega_i(t_1 - \tau)) u(\tau) d\tau, \end{cases} \quad i = 1, \dots, N. \tag{14}$$

Selecting $t_1 = t_0,\ u(t) = u_0(t_1 - t)$, one obtains the correct equality, which means that the system has the global controllability property due to the arbitrariness of $(q_1^*, -p_1^*, \dots, q_N^*, -p_N^*)$.

Remark 1. The controllable state $(q_1^*, p_1^*, \dots, q_N^*, p_N^*)$, from where the origin is reached with control $u(t)$ in time t_0, corresponds to the accessible state with the same control for the same time taken in reverse time.

4. MAXIMUM PRINCIPLE EQUATIONS

The statements of the Pontryagin's maximum principle are given to study the optimal control problem (1)–(4).

(a) The Hamiltonian for the optimal control problem

$$\begin{aligned} h_u &= -\lambda_0 + \xi_1 p_1 + \eta_1(-\omega_1^2 q_1 + u) + \ldots + \xi_N p_N + \eta_N(-\omega_N^2 q_N + u) \\ &= -\lambda_0 + \sum_{i=1}^{N} \xi_i p_i + \eta_i(-\omega_i^2 q_i + u), \quad -\lambda_0 > 0, \end{aligned} \tag{15}$$

where $\boldsymbol{\psi}(t) = (\xi_1(t), \eta_1(t), \dots, \xi_N(t), \eta_N(t))^{\mathrm{T}}$ is a vector of adjoint variables, and λ_0 is a constant.

(b) The Hamiltonian system including the equations of motion and the adjoint system of equations:

$$\begin{cases} \dot{\xi}_i(t) = -\dfrac{\partial h_u}{\partial q_i} = \omega_i^2 \eta_i(t), \\ \dot{\eta}_i(t) = -\dfrac{\partial h_u}{\partial p_i} = -\xi_i(t), \end{cases} \quad i = 1, \dots, N. \tag{16}$$

(c) Maximum condition:

$$h_{u^*} = \max_{u(\cdot) \in \mathbb{U}} h_u = -\lambda_0 + \sum_{i=1}^{N} (\xi_i p_i - \eta_i \omega_i^2 q_i) + u^* \sum_{i=1}^{N} \eta_i. \tag{17}$$

The adjoint system (16) and its solution with $2N$ unknown constant coefficients:

$$\begin{cases} \ddot{\eta}_1(t) = -\omega_1^2 \eta_1(t), \\ \cdots \\ \ddot{\eta}_N(t) = -\omega_N^2 \eta_N(t); \end{cases} \qquad \begin{cases} \eta_1 = C_1^1 \cos \omega_1 t + C_1^2 \sin \omega_1 t, \\ \cdots \\ \eta_N = C_N^1 \cos \omega_N t + C_N^2 \sin \omega_N t. \end{cases} \tag{18}$$

The optimal control u^* is determined by the maximum condition (17):

$$u^*(t) = \varepsilon \operatorname{sgn} \sum_{i=1}^{N} \eta_i = \varepsilon \operatorname{sgn} \left(\sum_{i=1}^{N} C_i^1 \cos \omega_i t + C_i^2 \sin \omega_i t \right). \tag{19}$$

In the next section, the necessary extremum conditions for a system of N nonsynchronous oscillators will be deduced, which will make it possible to study the solution of the problem in any given switching class of relay control (19).

5. THE NECESSARY EXTREMUM CONDITION

The control $u^*(t)$ has relay type according to (19). The control switchings occur at time instants t_m, $m = \overline{1, K-1}$. The duration of the nth control interval is denoted by τ_n, $n = \overline{1, K}$. Then $u^*(t)$ with $K-1$ switchings and $K \in \mathbb{N}$ control intervals has the form with accuracy up to the sign shown in Fig. 1. Such control belongs to the $(K-1)$-switching control class.

The solution for (1)–(3) can be written out for different values of K, taking into account the type of optimal control $u^*(t)$. The control on the first interval can be chosen either ε or $-\varepsilon$, therefore a parameter s equal to 0 and 1 is introduced, respectively.

$$\begin{cases} 2 \sum\limits_{j=1}^{K} (-1)^{j+1} \cos \left(\omega_i \sum\limits_{k=j}^{K} \tau_k \right) - \cos \left(\omega_i \sum\limits_{k=1}^{K} \tau_k \right) \\ = (-1)^{K-1} + (-1)^s \dfrac{\omega_i^2}{\varepsilon} \left(\dfrac{p_i^*}{\omega_i} \sin \omega_i T^0 + q_i^* \cos \omega_i T^0 \right), \\ 2 \sum\limits_{j=1}^{K} (-1)^{j+1} \sin \left(\omega_i \sum\limits_{k=j}^{K} \tau_k \right) - \sin \left(\omega_i \sum\limits_{k=1}^{K} \tau_k \right) \\ = (-1)^s \dfrac{\omega_i}{\varepsilon} \Big(- p_i^* \cos \omega_i T^0 + q_i^* \omega_i \sin \omega_i T^0 \Big), \end{cases} \quad i = 1, \ldots, N. \tag{20}$$

Remark 2. The system (20) of $2N$ equations makes it possible to investigate the $(2N-1)$-switching control class for which it is necessary to determine $2N$ interval durations.

Remark 3. The solution of the system (20) with control $u^*(t)$ for the initial vector $\mathbf{x}_0 = (q_1^*, p_1^*, \ldots, q_N^*, p_N^*)^{\mathrm{T}}$ corresponds to $-u^*(t)$ for $-\mathbf{x}_0 = (-q_1^*, -p_1^*, \ldots, -q_N^*, -p_N^*)^{\mathrm{T}}$.

For the investigation of control classes with a large number of switchings, the following necessary extremum conditions are given, similar to the conditions for two oscillators obtained in [17].

Lemma 3 (Necessary Extremum Condition). *Any solution of the problem* (1)–(4) *in the class of piecewise continuous controls* (19) *satisfies both the system of equations* (20) *and the additional*

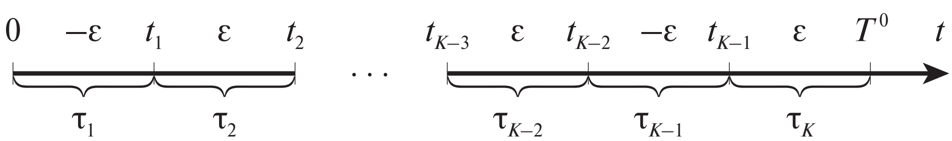

Fig. 1. Form of optimal control $u^*(t)$.

K − 2*N* *equations:*

$$\det\begin{pmatrix} \cos(\omega_1 t_s) & \cos(\omega_1 t_{s+1}) & \dots & \cos(\omega_1 t_{s+2N-1}) \\ \sin(\omega_1 t_s) & \sin(\omega_1 t_{s+1}) & \dots & \sin(\omega_1 t_{s+2N-1}) \\ \dots & \dots & \ddots & \dots \\ \cos(\omega_N t_s) & \cos(\omega_N t_{s+1}) & \dots & \cos(\omega_N t_{s+2N-1}) \\ \sin(\omega_N t_s) & \sin(\omega_N t_{s+1}) & \dots & \sin(\omega_N t_{s+2N-1}) \end{pmatrix} = 0, \qquad s = 1, \dots, K-2N. \tag{21}$$

Proof. The optimal relay control switching at instant t_m, $m = 1, \dots, K-1$ leads to zero (19):

$$\sum_{i=1}^{N} C_i^1 \cos\omega_i t_m + C_i^2 \sin\omega_i t_m = 0,\ m = 1, \dots, K-1. \tag{22}$$

For compactness, let us write all equations of (22) in matrix form, as follows,

$$(\mathbf{C}, \mathbf{\Omega}_m) = 0, \quad m = 1, \dots, K-1, \tag{23}$$

where

$$\mathbf{C} = \left(C_1^1,\ C_1^2,\ \dots\ , C_N^1,\ C_N^2\right),\ \mathbf{\Omega}_m = \begin{pmatrix} \cos(\omega_1 t_m) \\ \sin(\omega_1 t_m) \\ \dots \\ \cos(\omega_N t_m) \\ \sin(\omega_N t_m) \end{pmatrix}.$$

The obtained equations (23) will be reformulated in the following form:

$$\begin{pmatrix} \cos(\omega_1 t_1) & \cos(\omega_1 t_2) & \dots & \cos(\omega_1 t_{K-2}) & \cos(\omega_1 t_{K-1}) \\ \sin(\omega_1 t_1) & \sin(\omega_1 t_2) & \dots & \sin(\omega_1 t_{K-2}) & \sin(\omega_1 t_{K-1}) \\ \dots & \dots & \ddots & \dots & \dots \\ \cos(\omega_N t_1) & \cos(\omega_N t_2) & \dots & \cos(\omega_N t_{K-2}) & \cos(\omega_N t_{K-1}) \\ \sin(\omega_N t_1) & \sin(\omega_N t_2) & \dots & \sin(\omega_N t_{K-2}) & \sin(\omega_N t_{K-1}) \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} C_1^1 \\ C_1^2 \\ \dots \\ C_N^1 \\ C_N^2 \end{pmatrix} = \vec{0}. \tag{24}$$

The existence condition of the nontrivial vector $\mathbf{C}$ is equivalent to $K-2N$ equations that look like

$$\det\left(\mathbf{\Omega}_s, \mathbf{\Omega}_{s+1}, \dots, \mathbf{\Omega}_{s+2N-1}\right) = 0, \quad s = 1, \dots, K-2N. \tag{25}$$

In the next section, an approach for finding the optimal solution based on the Neustadt–Eaton method will be given.

6. THE NUMERICAL ALGORITHM FOR FINDING THE INITIAL ADJOINT VECTOR

The unknown constant coefficients $\mathbf{C}$ included in the optimal control (19) are explicitly computed from the initial adjoint vector $\boldsymbol{\psi}(0) = (\xi_1(0), \eta_1(0), \dots, \xi_N(0), \eta_N(0))^{\mathrm{T}}$ by the following rule:

$$\mathbf{C} = \left(C_1^1,\ C_1^2,\ \dots,\ C_N^1,\ C_N^2\right) = \left(\eta_1(0),\ -\frac{\xi_1(0)}{\omega_1},\ \dots,\ \eta_N(0),\ -\frac{\xi_N(0)}{\omega_1}\right). \tag{26}$$

A set H_* is introduced which consists of vectors $\boldsymbol{\psi}(0)$ and defines trajectories $\mathbf{x}(t)$ of a system (1) with boundary conditions (3).

The method of finding $\boldsymbol{\psi}(0)$, based on the Neustadt–Eaton iterative algorithm, is given as a sequence of steps [3].

Initialization. The fixed initial vector $\mathbf{x}_0$ is set. One chooses the initial adjoint normalized vector $\boldsymbol{\psi}_0 = (\xi_1^{(0)}, \eta_1^{(0)}, \dots, \xi_N^{(0)}, \eta_N^{(0)})^{\mathrm{T}} = -\frac{\mathbf{x}_0}{|\mathbf{x}_0|}$.

Remark 4. The initial vector $\boldsymbol{\psi}_0$ can be any vector located in the half-space D, which is defined by the hyperplane orthogonal to the vector $\mathbf{x}_0$.

The following variables are entered:

$$I_{F(\boldsymbol{\psi}_h)}(\boldsymbol{\psi}_h) = \begin{pmatrix} \int\limits_0^{F(\boldsymbol{\psi}_h)} \left(\frac{\sin\omega_1\tau}{\omega_1}\right)\varepsilon\,\mathrm{sgn}\left(\sum_{i=1}^N \eta_i^{(h)}\cos\omega_i\tau - \frac{\xi_i^{(h)}}{\omega_i}\sin\omega_i\tau\right)d\tau \\ -\int\limits_0^{F(\boldsymbol{\psi}_h)} (\cos\omega_1\tau)\,\varepsilon\,\mathrm{sgn}\left(\sum_{i=1}^N \eta_i^{(h)}\cos\omega_i\tau - \frac{\xi_i^{(h)}}{\omega_i}\sin\omega_i\tau\right)d\tau \\ \cdots \\ \int\limits_0^{F(\boldsymbol{\psi}_h)} \left(\frac{\sin\omega_N\tau}{\omega_N}\right)\varepsilon\,\mathrm{sgn}\left(\sum_{i=1}^N \eta_i^{(h)}\cos\omega_i\tau - \frac{\xi_i^{(h)}}{\omega_i}\sin\omega_i\tau\right)d\tau \\ -\int\limits_0^{F(\boldsymbol{\psi}_h)} (\cos\omega_N\tau)\,\varepsilon\,\mathrm{sgn}\left(\sum_{i=1}^N \eta_i^{(h)}\cos\omega_i\tau - \frac{\xi_i^{(h)}}{\omega_i}\sin\omega_i\tau\right)d\tau \end{pmatrix}, \tag{27}$$

where $\boldsymbol{\psi}_h = (\xi_1^{(h)}, \eta_1^{(h)}, \ldots, \xi_N^{(h)}, \eta_N^{(h)})^{\mathrm{T}}$ is the vector constructed at the hth ($h = 0, 1, \ldots$) step of the algorithm. $F(\boldsymbol{\psi}_h)$ is defined as the solution of the equation

$$\langle \boldsymbol{\psi}_h,\ \mathbf{x}_0 - I_{F(\boldsymbol{\psi}_h)}(\boldsymbol{\psi}_h)\rangle = 0. \tag{28}$$

Step 1. The following vector $\boldsymbol{\psi}_1$ is defined in accordance with the equations:

$$\begin{gathered} \widetilde{\boldsymbol{\psi}}_1^{(m)} = \boldsymbol{\psi}_0 - 2^{-m}(\mathbf{x}_0 - I_{F(\boldsymbol{\psi}_0)}(\boldsymbol{\psi}_0)), \\ \boldsymbol{\psi}_1 = \frac{\widetilde{\boldsymbol{\psi}}_1^{(m)}}{|\widetilde{\boldsymbol{\psi}}_1^{(m)}|}, \end{gathered} \tag{29}$$

at the same time the smallest non-negative integer m is chosen for which the vector $\boldsymbol{\psi}_1$ satisfies the inequality

$$\langle \boldsymbol{\psi}_1, \mathbf{x}_0 - I_{F(\boldsymbol{\psi}_0)}(\boldsymbol{\psi}_1)\rangle < -\frac{|\mathbf{x}_0 - I_{F(\boldsymbol{\psi}_0)}(\boldsymbol{\psi}_0)|^2}{2^{m+1}}. \tag{30}$$

Step 2. Assuming that the vectors $\boldsymbol{\psi}_0,\ \boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_h$ are constructed inductively in the same area, the following vector $\boldsymbol{\psi}_{h+1}$ is defined as

$$\begin{gathered} \widetilde{\boldsymbol{\psi}}_{h+1}^{(m)} = \boldsymbol{\psi}_h - 2^{-m}(\mathbf{x}_0 - I_{F(\boldsymbol{\psi}_h)}(\boldsymbol{\psi}_h)), \\ \boldsymbol{\psi}_{h+1} = \frac{\widetilde{\boldsymbol{\psi}}_{h+1}^{(m)}}{|\widetilde{\boldsymbol{\psi}}_{h+1}^{(m)}|}, \end{gathered} \tag{31}$$

at the same time the smallest non-negative integer m is chosen for which the vector $\boldsymbol{\psi}_{h+1}$ satisfies the inequality

$$\langle \boldsymbol{\psi}_{h+1}, \mathbf{x}_0 - I_{F(\boldsymbol{\psi}_h)}(\boldsymbol{\psi}_{h+1})\rangle < -\frac{|\mathbf{x}_0 - I_{F(\boldsymbol{\psi}_h)}(\boldsymbol{\psi}_h)|^2}{2^{m+1}}. \tag{32}$$

Then either $\boldsymbol{\psi}_h$ is in H_* for some h, or the construction leads to an infinite sequence of vectors $\boldsymbol{\psi}_0, \boldsymbol{\psi}_1, \boldsymbol{\psi}_2, \ldots$ lying in the halfspace D and having the following properties:

1) the numbers $F(\boldsymbol{\psi}_0),\ F(\boldsymbol{\psi}_1),\ F(\boldsymbol{\psi}_2), \ldots$ form a monotonically increasing sequence converging to the number T^0;
2) $\lim\limits_{h\to\infty} I_{F(\boldsymbol{\psi}_h)}(\boldsymbol{\psi}_h) = \mathbf{x}_0$;
3) the sequence of vectors $\boldsymbol{\psi}_0, \boldsymbol{\psi}_1, \boldsymbol{\psi}_2, \ldots$ is close to the set H_*.

Remark 5. About finding (28) solution

The function $\langle \boldsymbol{\psi}_h,\ \mathbf{x}_0 - I_t(\boldsymbol{\psi}_h)\rangle$ is continuous on the variables $\boldsymbol{\psi}_h, t$. It increases monotonically along t [3] for any fixed $\boldsymbol{\psi}_h$. Therefore, the solution of (28) is determined by constructing the values of the above function for the increasing sequence $F(\boldsymbol{\psi}_h)$ using the bisection method and numerical integration.

7. MODELING

To demonstrate the results obtained in Sections 5, 6, the optimal control for two nonsynchronous oscillators ($\omega_1 = 1,\ \omega_2 = 1.4$) using both the necessary extremum conditions and the Neustadt–Eaton method is calculated. Here and below, the value $\varepsilon = 0.4$ is chosen for the control constraint.

The set of initial states $\mathbf{x}_0 = (q_1^*, p_1^*, 0, 0)^{\mathrm{T}}$ is considered, where $|q_1^*| < 2.2$, $|p_1^*| < 1.5$ (controllability set). For the three–($K = 4$) and four–($K = 5$) switching control classes, solutions are searched according to the 3 Lemma. Each obtained solution is checked for conformity with the originally proposed switching class. In accordance with the Remark 3, it is necessary to investigate only half of the considered area. The classified plane of the first oscillator, which also includes the two-switching class whose analytic description can be found in [22], is shown in Fig. 2.

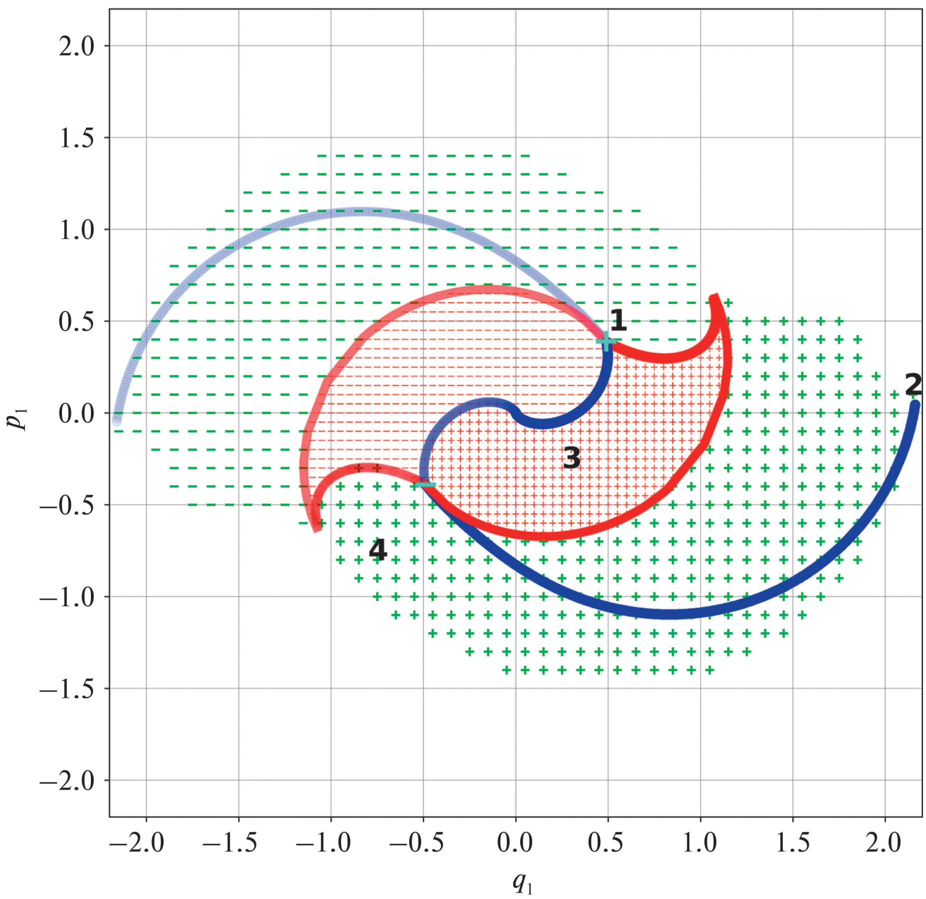

Fig. 2. Controllability set of the first oscillator (Lemma 3).

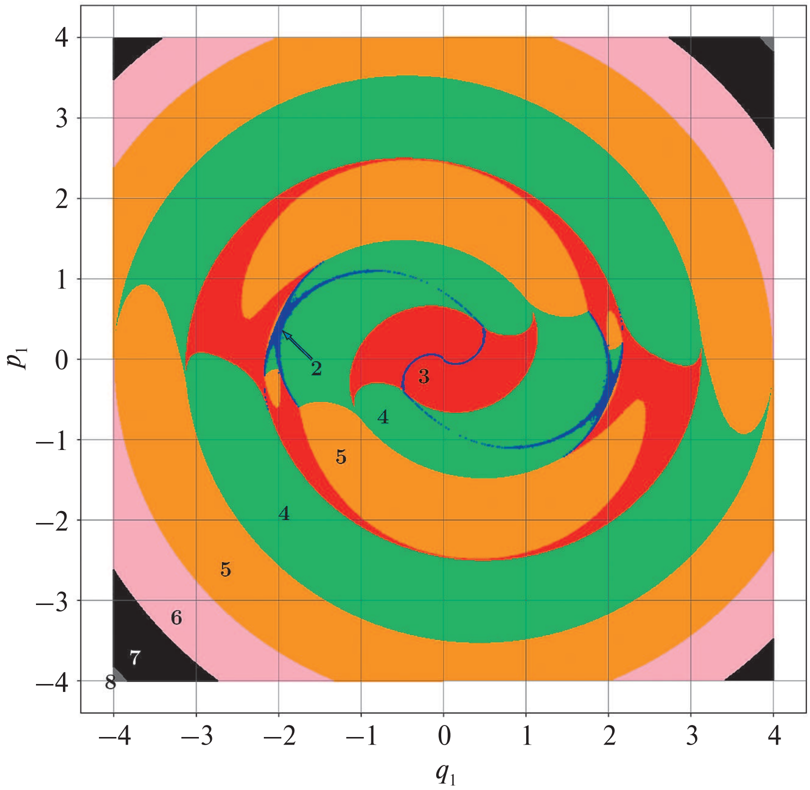

Fig. 3. Controllability set of the first oscillator (Neustadt–Eaton method).

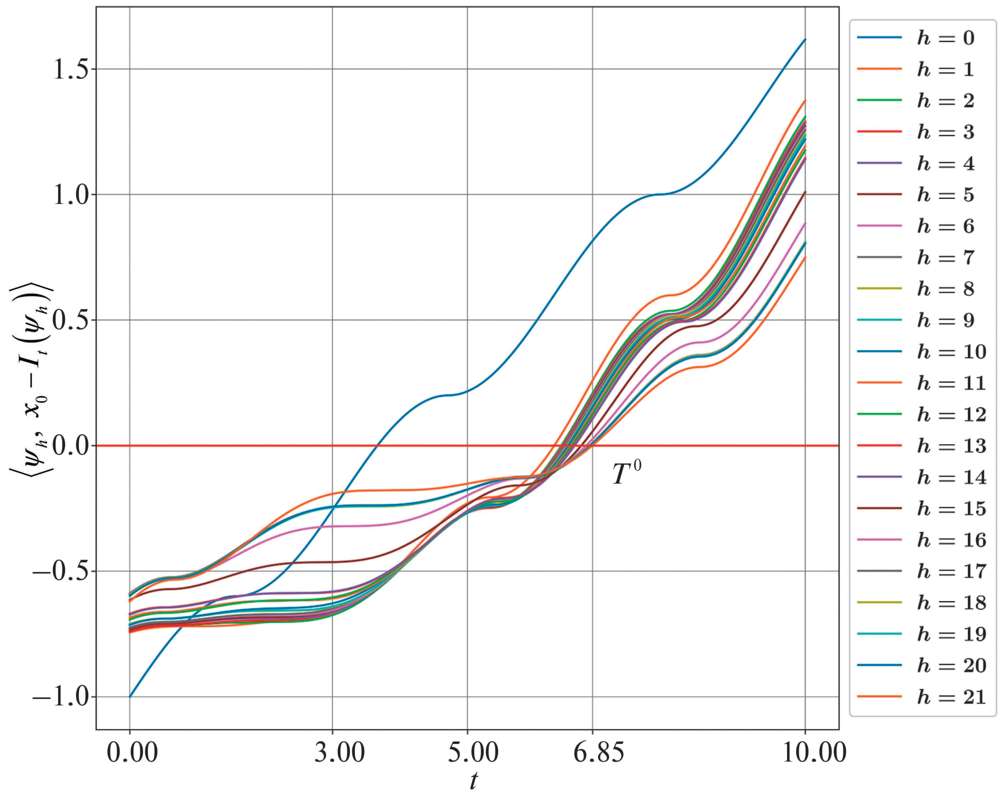

Fig. 4. The graph of the function (28) for different h values.

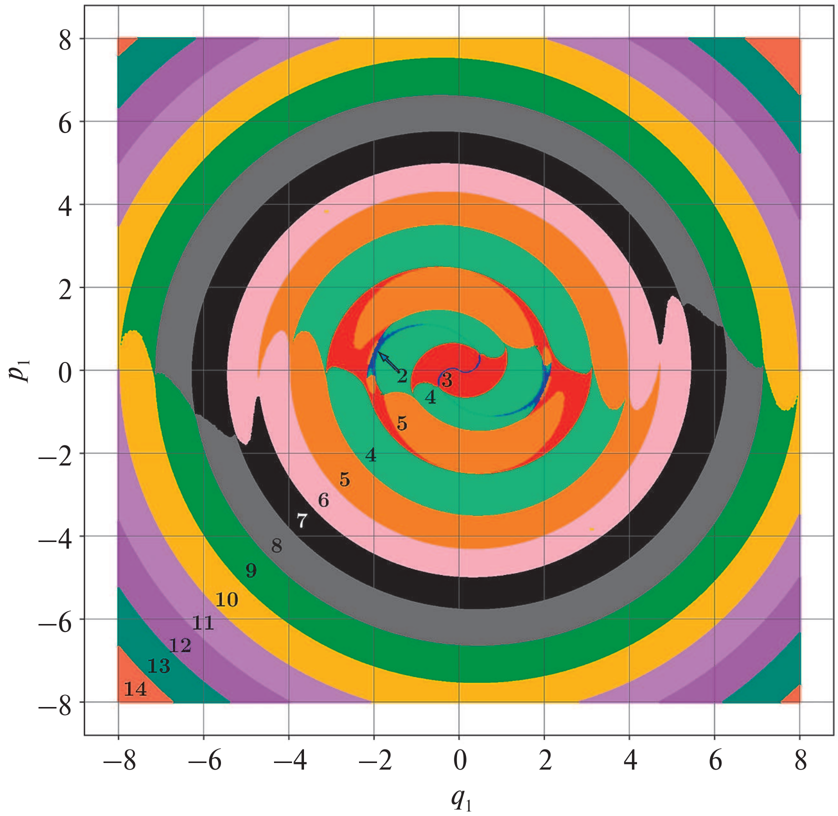

Fig. 5. Controllability set of the first oscillator.

The number of control switchings is marked on the graph with numbers. The plus and minus markers correspond to the controls with values ε and $-\varepsilon$ on the first interval, respectively. The two-switching class curves with high transparency match the controls taking negative values on the first interval. The following features of the proposed approach are noted: the lack of information about the admissibility of using a particular control class for a given initial condition, which leads to the necessarity of investigation different K, the complexity of finding a solution to the system of nonlinear equations of order K. However, the above method allows us to study various degenerate cases, such as the two-switching control class, and to obtain analytic constructions for class boundaries, such as the blue and red curves in Fig. 2.

The classified plane of the first oscillator obtained with the Neustadt–Eaton method is given in Fig. 3. The main parameters in finding a new approximation $\boldsymbol{\psi}_h$ were the root determination accuracy of the equation (28) and a constraint on the error, defined as the Euclidean distance from the end of the found trajectory to the origin. Each new value of $F(\boldsymbol{\psi}_h)$ is used as an initial approximation to determine $F(\boldsymbol{\psi}_{h+1})$ due to monotonically increasing numbers $F(\boldsymbol{\psi}_0),\ F(\boldsymbol{\psi}_1),\dots$. The above is illustrated in Fig. 4, which shows the graph of the function $\langle \boldsymbol{\psi}_h,\ \mathbf{x}_0 - I_t(\boldsymbol{\psi}_h)\rangle$ for different h when the initial vector $\mathbf{x}_0 = (0, 1, 0, 0)^{\mathrm{T}}$ is selected.

In Fig. 5 we analyze the larger controllability set, which includes 14 control classes considering degenerate classes of one and two switchings. The number of control switchings is also marked on the graph with numbers.

An example for four nonsynchronous oscillators with the following parameters is given to illustrate the workings of the Neustadt–Eaton method:

$$\varepsilon = 0.4, \quad \{\omega_i\}_{i=1}^{4} = \{1,\ 1.4,\ 1.7,\ 2\}, \quad \mathbf{x}_0 = (2,\ 2.5,\ 0,\ \dots,\ 0,\ 0)^{\mathrm{T}}.$$

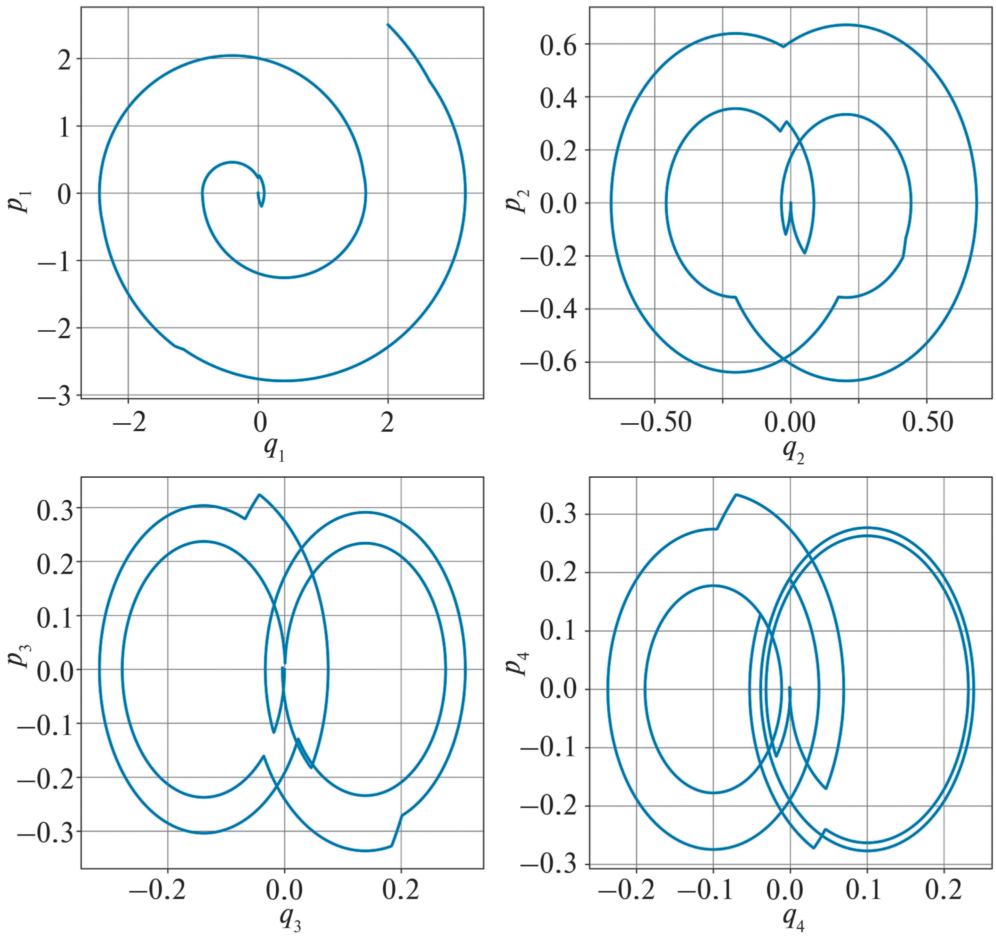

Fig. 6. Phase planes of the four oscillators.

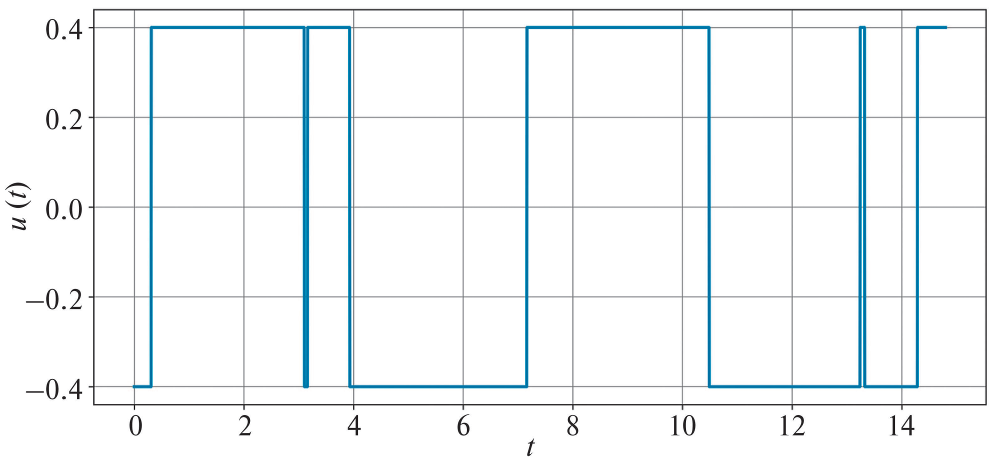

Fig. 7. Scalar control with nine switchings for four oscillators.

The results of the algorithm are: the initial vector of the adjoint system

$$\boldsymbol{\psi}(0) = (-0.336, -0.372, -0.334, -0.149, -0.504, -0.158, -0.562, 0.141)$$

and the problem criterion $T^0 = 14.795$. The motion trajectories of the oscillators are shown in Fig. 6 with a scalar control illustrated in Fig. 7.

The program implementation of the Neustadt–Eaton algorithm is fast and efficient, giving an approximate classification of the controllability set of the first oscillator according to the number of control switchings.

8. CONCLUSION

The time-optimal control problem for a group of nonsynchronous oscillators with a limited scalar control is considered. The relation between accessibility and controllability sets is demonstrated. The trajectories are found to bring a group of oscillators to the origin by using the necessary extremum conditions and the Neustadt–Eaton iterative algorithm. The obtained trajectory classifications based on the number of relay control switchings are compared. The results of the iterative algorithm, such as switching control class, estimation of the system motion time and switching moments, can be used as an initial approximation to find a solution based on the necessary extremum conditions.

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